Lecture 29

Cofactor Expansion

- The ij-minor of $A \in M_{n \times n}$ is the matrix $A_{ij} \in M_{(n-1)\times(n-1)}$ obtained from A by deleting the i^{th} row and j^{th} column.
- The ij-cofactor of A is the number $C_{ij} = (-1)^{i+j} \det[A_{ij}]$

$\boxed{\text{Ex}}$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 4 & 5 & 9 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 0 & 7 \\ 4 & 9 \end{bmatrix}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 7 \\ 4 & 9 \end{vmatrix}$$
$$= (-1)\left[(0)(9) - 7(4)\right]$$
$$= 28$$

- **Theorem** $\forall\ A = (a_{ij}) \in M_{n \times n}$ and any $i \in \{1, \ldots, n\}$ we have

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \underbrace{a_{i1} C_{i1} + \ldots + a_{in} C_{in}}$$

cofactor expansion of $\det A$ about the i^{th} row.

- Similary if $j \in \{1, \ldots, n\}$ then $\det A = \sum_{i=1}^{n} a_{ij} C_{ij}$

$\boxed{\text{Ex}}$ Fix $i=2$

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 4 & 5 & 9 \end{vmatrix} = 0\cdot(-1)^{2+1}\begin{vmatrix} 2 & -3 \\ 5 & 9 \end{vmatrix} + 1\cdot(-1)^{2+2}\begin{vmatrix} 1 & -3 \\ 4 & 9 \end{vmatrix} + 7(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 0 + 1(9+12) - 7(5-8)$$

$$= 42$$

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 4 & 5 & 9 \end{vmatrix} \qquad \det A = (-3)\cdot 1 \cdot \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} + 7\cdot(-1)\cdot\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} + 9(1)\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= (-3)(-4) + (-7)(-3) + (9)(1) = 42$$

$\boxed{\text{Ex}}$

$$A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ -1 & -3 & -7 & -28 \end{vmatrix}$$

$$\det A = ? = 3\cdot(1)\begin{vmatrix} 5 & 16 & 8 \\ 9 & 10 & 12 \\ -1 & -7 & -28 \end{vmatrix} + \dots$$

$\boxed{\text{Ex}}$

$$\begin{vmatrix} 3 & 0 & 1 & 4 \\ 0 & 1 & 7 & 0 \\ 10 & 0 & 2 & 0 \\ 18 & 0 & 0 & -11 \end{vmatrix} = 18 \cdot (-1) \begin{vmatrix} 0 & 1 & 4 \\ 1 & 7 & 0 \\ 0 & 2 & 0 \end{vmatrix}$$

$$+ 0 \cdot (*)$$

$$+ 0 \cdot (*)$$

$$+ (-1) \cdot (1) \cdot \begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & 7 \\ 10 & 0 & 2 \end{vmatrix}$$

$\boxed{\text{Ex}}$

$$\begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} = e^{2x} \cos^2 y - (-e^{2x} \sin^2 y)$$

$$= e^{2x}$$

- Thm guarantees that a determinant f_n exists
- Cofactor expansion of $\det(A)$ about row i = cofactor expansion of $\det(A^T)$ about column j
- Corollary: $\det(A) = \det(A^T)$

- Cofactor expansion implies many wierd formulas involving determinants:

  - Given $A \in M_{n\times n}$, Let $C \in M_{n\times n}$ be the matrix whose ij-entry is the ij-cofactor C_{ij} of A.

↳ Theorem: If A is invertible and C is its co-factor matrix for A, then $A^{-1} = \frac{C^T}{\det(A)}$

Proof: Suffices to show $A \cdot \frac{C^T}{\det(A)} = I \iff A \cdot C^T = (\det A) I$

$$(A \cdot C^T)_{ij} = \left[(\det A) I\right]_{ij}$$

ij-entry of $A \cdot C^T = (\text{row}_i \text{ of } A) \cdot (\text{row}_j \text{ of } C)$

- if $i=j$: $(A \cdot C^T)_{ij} = a_{i1}C_{i1} + \dots + a_{in}C_{in} = \det(A)$

- if $i \neq j$: $(A \cdot C^T)_{ij} = a_{i1} + C_{j1} + \dots + a_{in}C_{jn} = \det \tilde{A}$

  ↳ where $\tilde{A}$ is the matrix obtained from A by j^{th} row of A with the i^{th} row of A. → $\det(\tilde{A}) = 0$ (because linearly dependent) ∎